Optimal Search Algorithm

Faculty of DS & AI
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Business AI Lab

2025-09

Content

- Optimal search
  - Definition
  - Greedy search
  - A* search
  - Properties of Heuristic Function

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Optimal Search

In practice problems

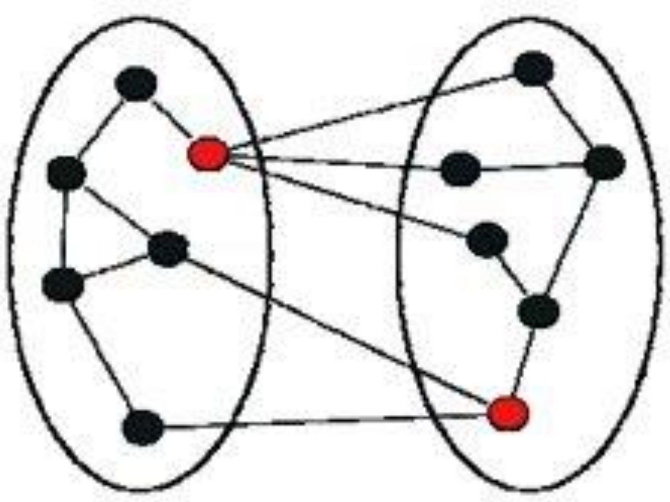
- We are often not only interested in finding a solution, but also whether the solution is optimal.
- Example:
    - Shortest path finding: consider the path cost..
    - 8-puzzle: consider the minimum number of moves to reach the goal.

👉 **In uninformed search and informed (heuristic) search, we have not yet considered path length or cost.**

Optimal Search

Graph Partition Problem:

- Given a graph, divide it into nnn equal-sized subsets such that the number of edges between subsets is minimized.



Optimal Search

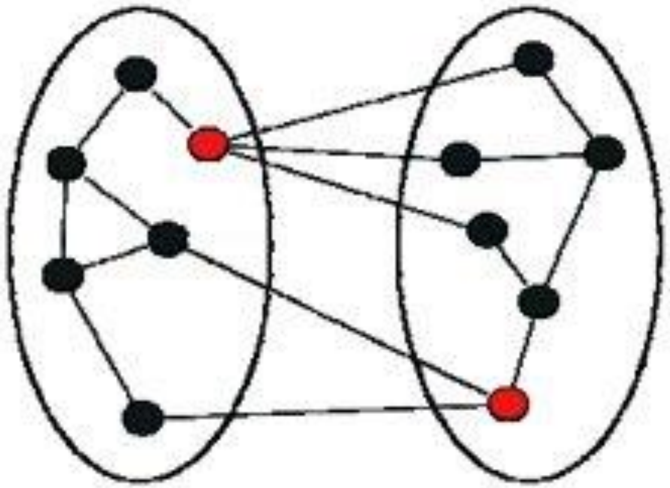
Graph Partition Problem:

- Given a graph, divide it into nnn equal-sized subsets such that the number of edges between subsets is minimized.

  - Each partition $G(V,E) \rightarrow \{G_1(V_1,E_1),G_2(V_2,E_2)\}$ is a **state**. A state can be represented by a **binary array**:
    - 0 -> vertex in group 1.
    - 1 -> vertex in group 2.
  - Example: we have state: u = $[0100011011]$
    - Group 1: $\{1,3,4,5,8\}$
    - Group 2: $\{2,6,7,9,10\}$
  - Evaluation function:
    - $F(u) = |V_1 - V_2| +$ number of cross edges (connected edge)
    - $|V_1 - V_2|$: balance term (equal partition).
    - Cross edges: edges between different groups.

❿ 👉 The goal is to find u* with **minimum $F(u)$**

❿ 👉 Optimal search = finding state u such that **f(u) is minimized**.



Optimal Search

Compare with Heuristic search

Criteria	**Heuristic Search**	**Optimal Search**
Evaluation	Based on heuristic $h(n)$	Based on total cost $g(n) + h(n)$
Goal	Find a solution quickly	Find the best (optimal) solution
Optimal guarantee	❌ No	✅ Yes (if conditions hold)
Example	Greedy Best-First Search	A*, Branch-and-bound search

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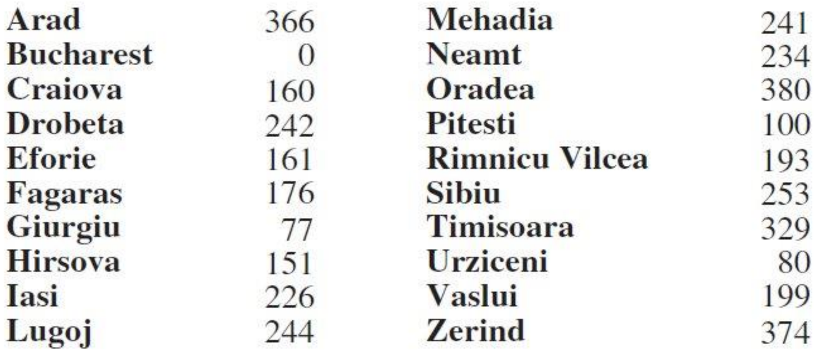
Optimal Search

Romania road map (textbook)

Arad -> Bucharest



Arad	366	Mehadia	241
Bucharest	0	Neamt	234
Craiova	160	Oradea	380
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Figure 3.16 Values of h_{SLD}—straight-line distances to Bucharest.

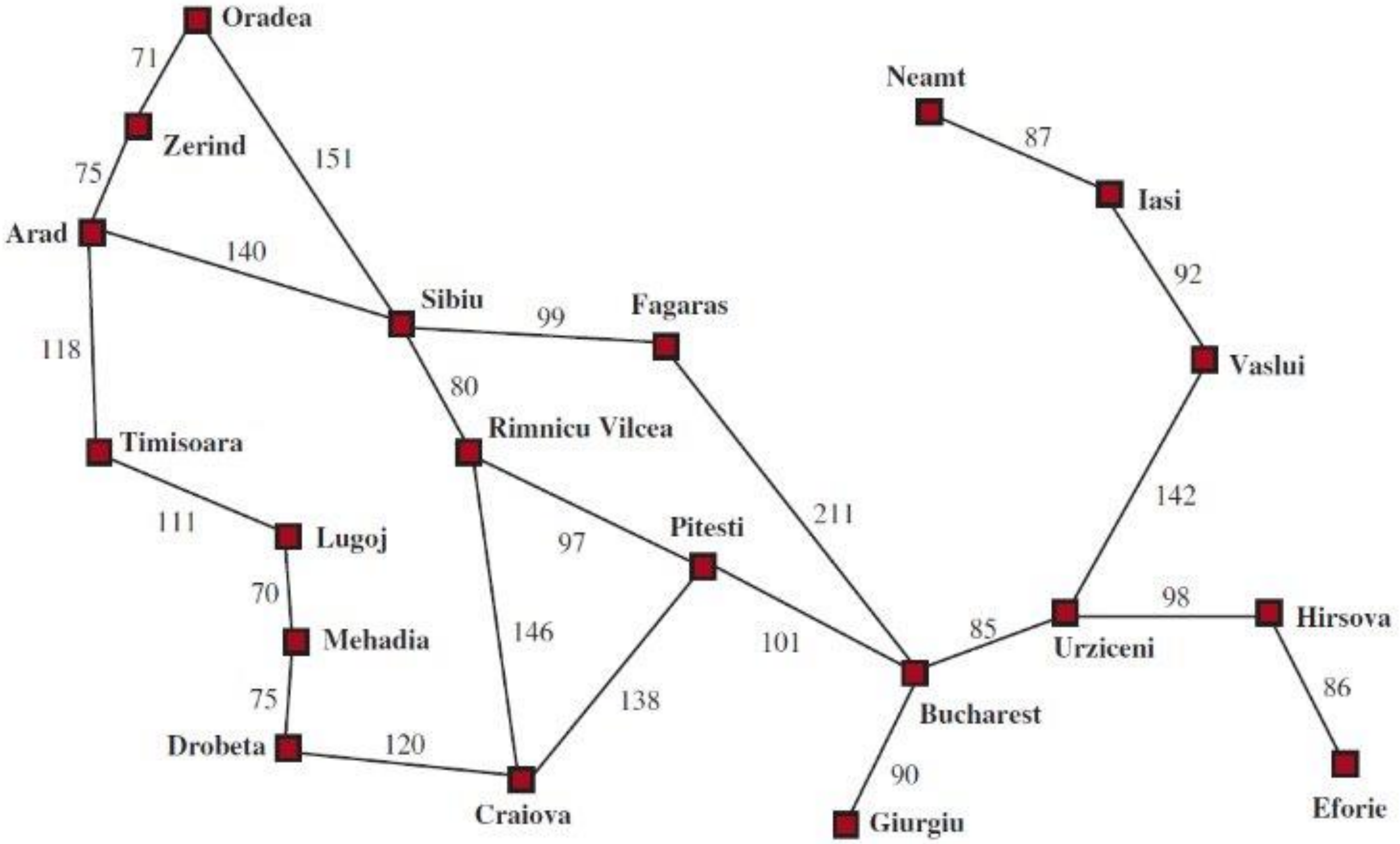


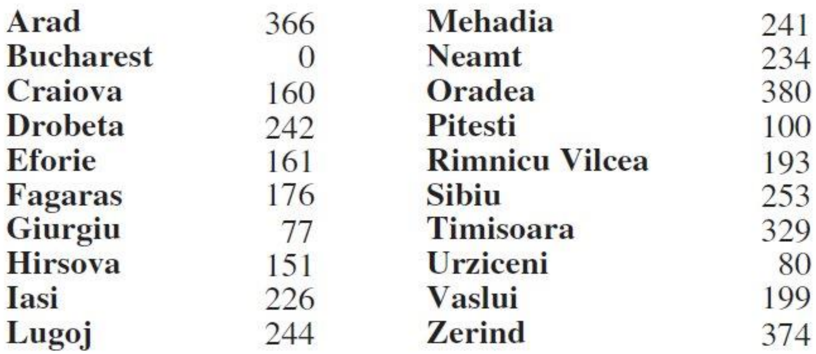
Figure 3.1 A simplified road map of part of Romania, with road distances in miles.

Optimal Search

Romania road map (textbook)

Arad -> Bucharest

h(u)



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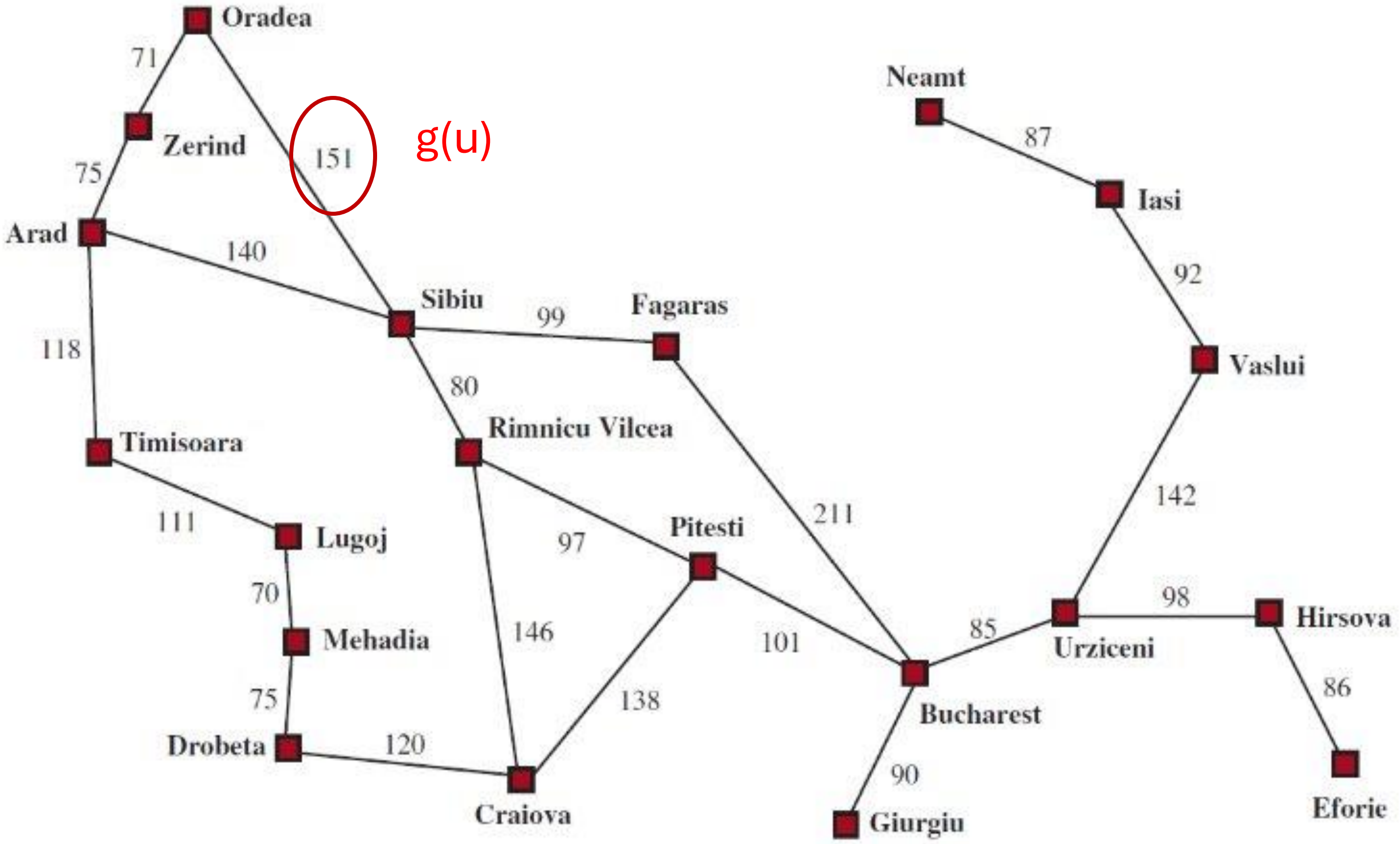

g(u)

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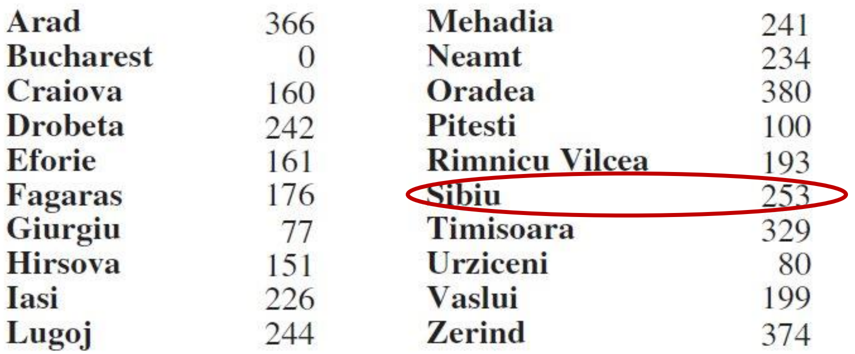
Optimal Search

Romania road map (textbook)

Arad -> Bucharest

h(u)



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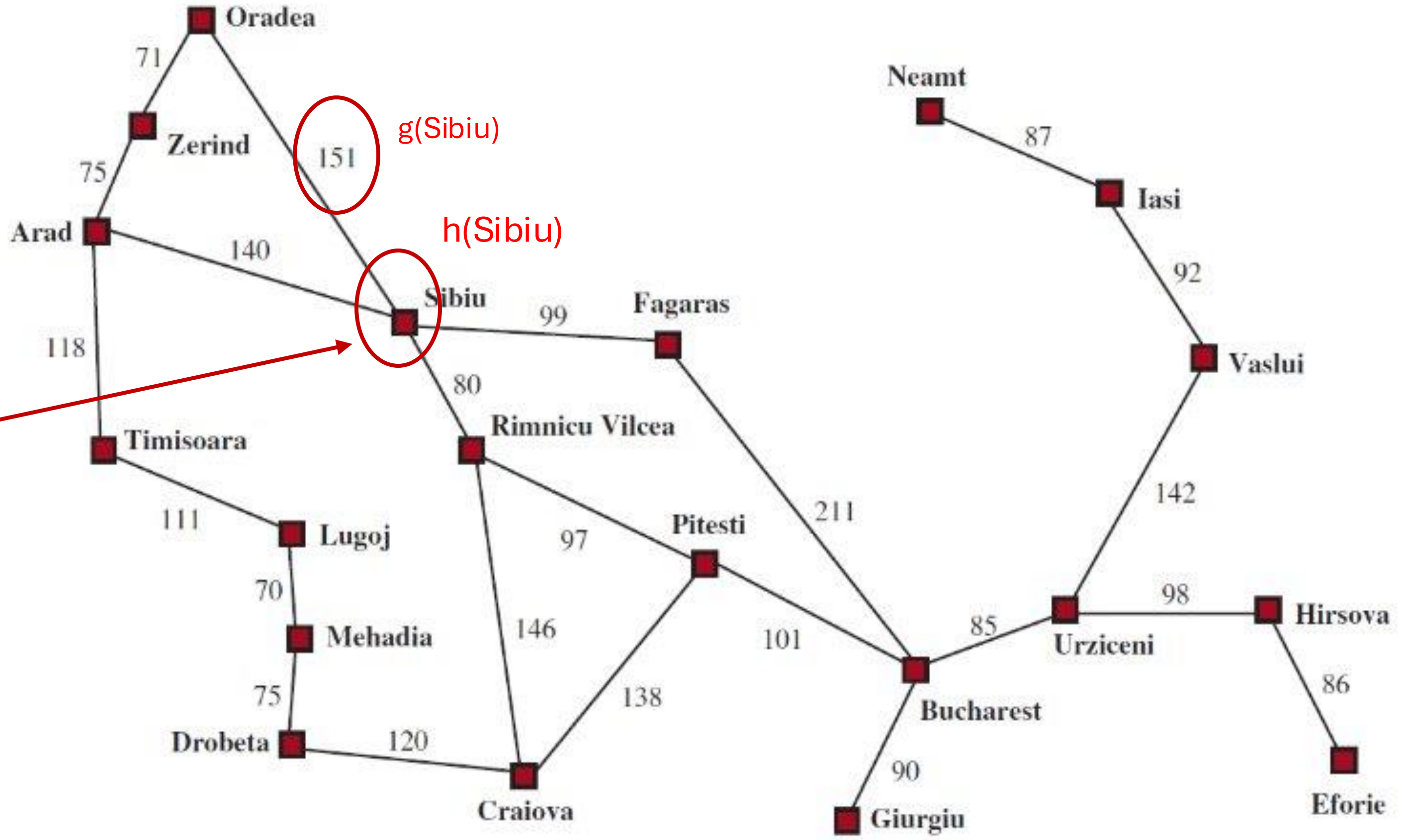

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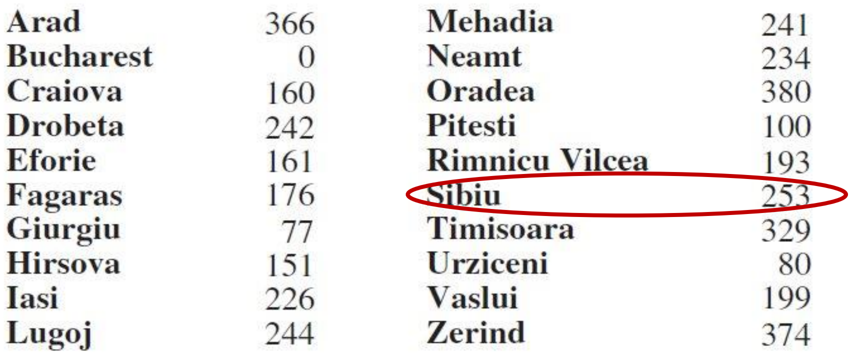
Optimal Search

Greedy search

h(u)



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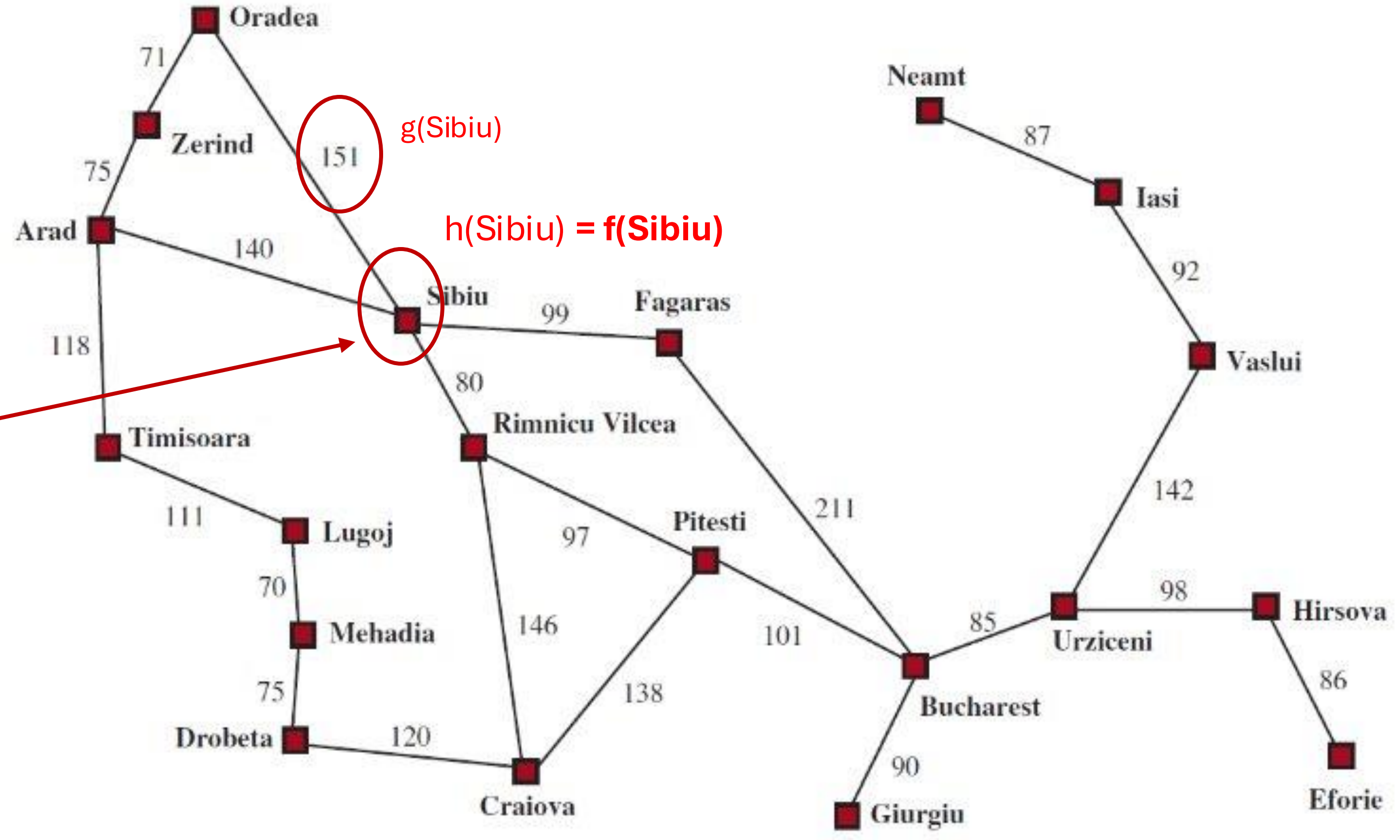


g(Sibiu)

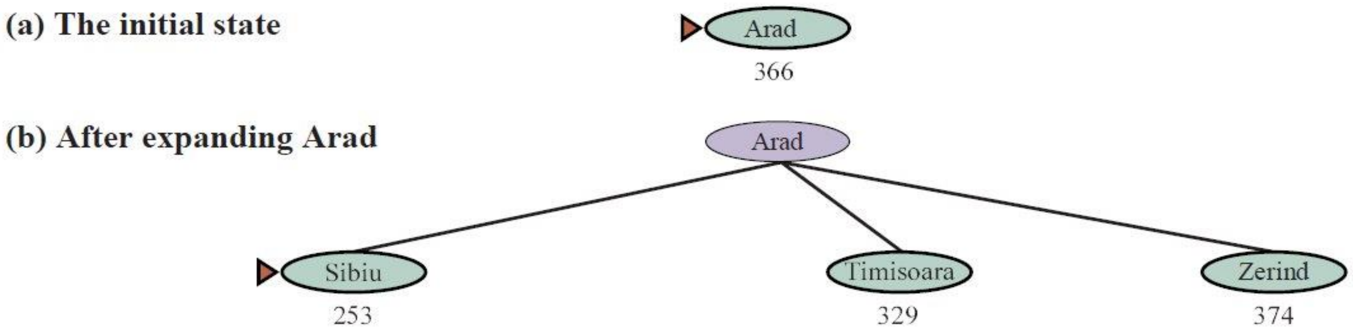
h(Sibiu) = **f(Sibiu)**

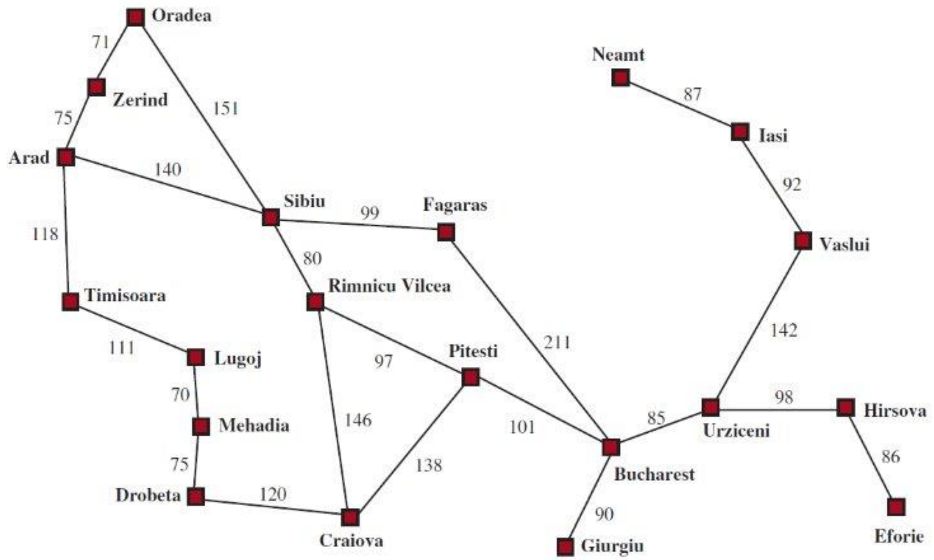
Figure 3.1 A simplified road map of part of Romania, with road distances in miles.

- select the node with the minimum value of h(u)
- hill-climbing search

Optimal Search

Greedy search





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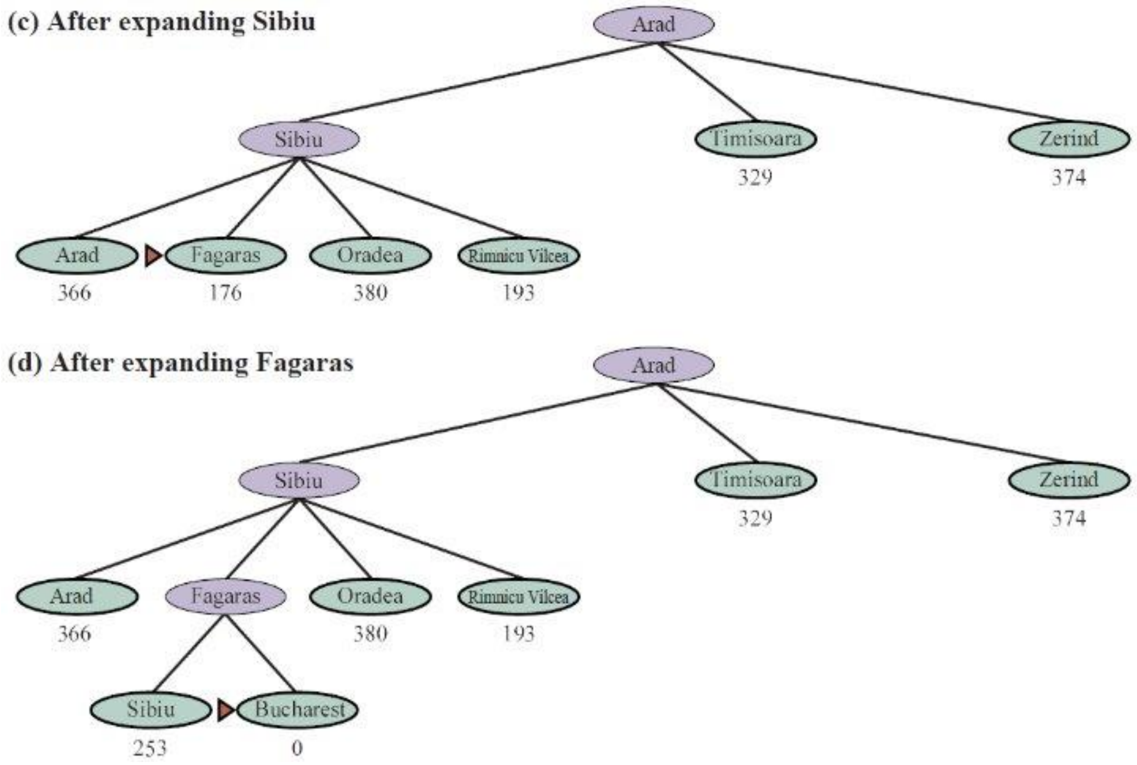
Optimal Search

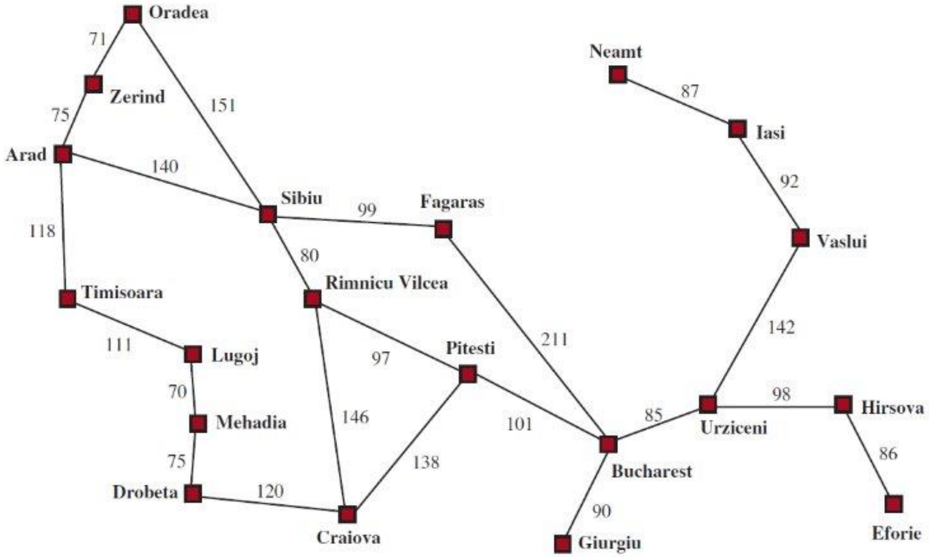
Greedy search



(c) After expanding Sibiu

(d) After expanding Fagaras

Path cost for the solution =
140+99+211 = 450 miles



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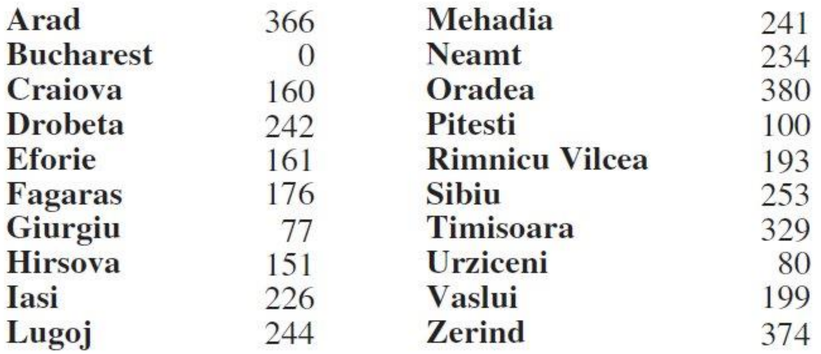

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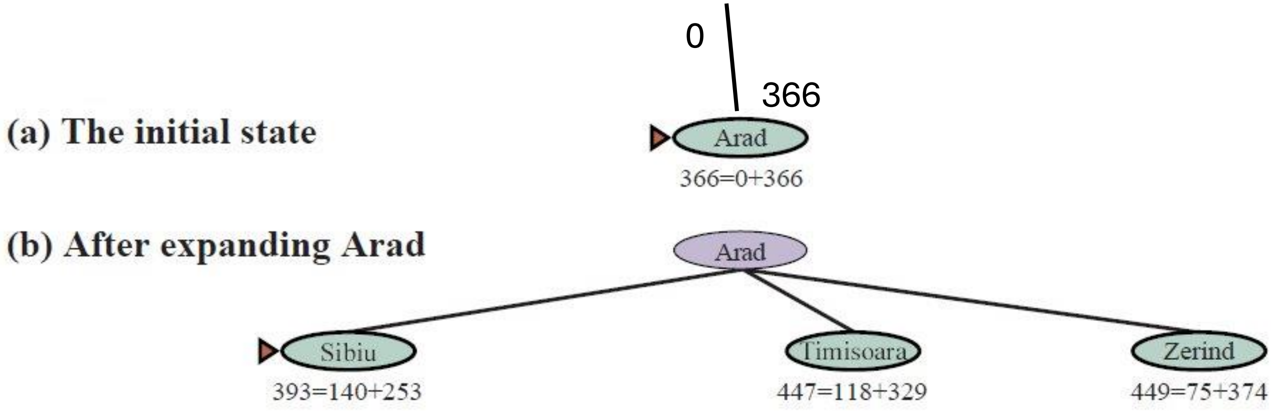
Optimal Search

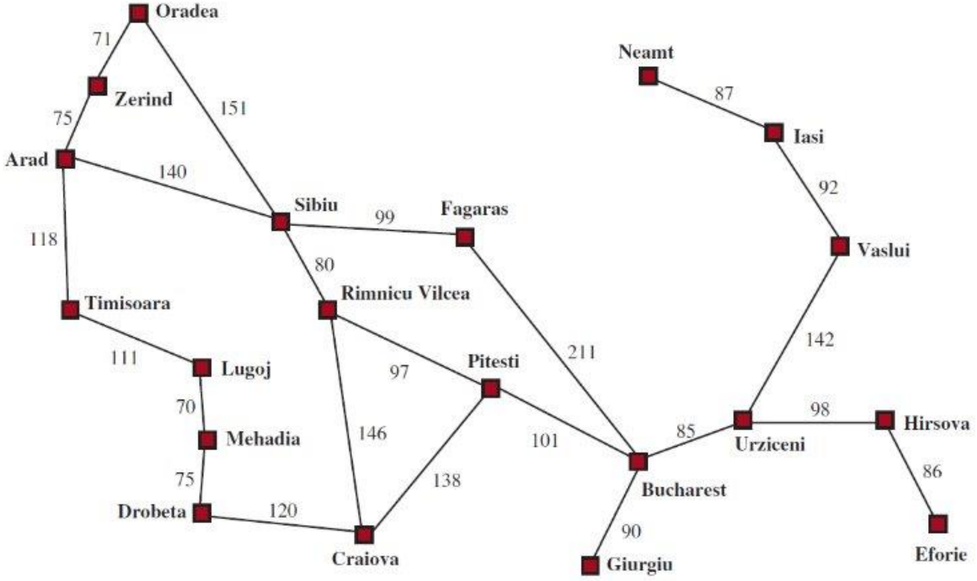
A* search

- select the node with the minimum value of

$$f(n) = g(n) + h(n)$$

$$f(n) = \text{ estimated cost of the cheapest solution through } n$$





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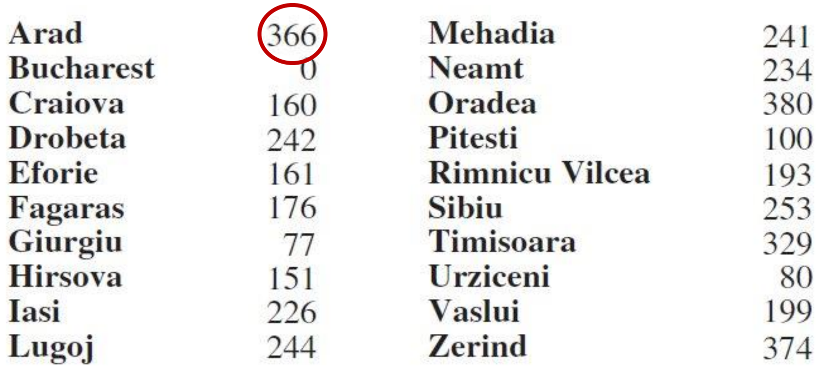

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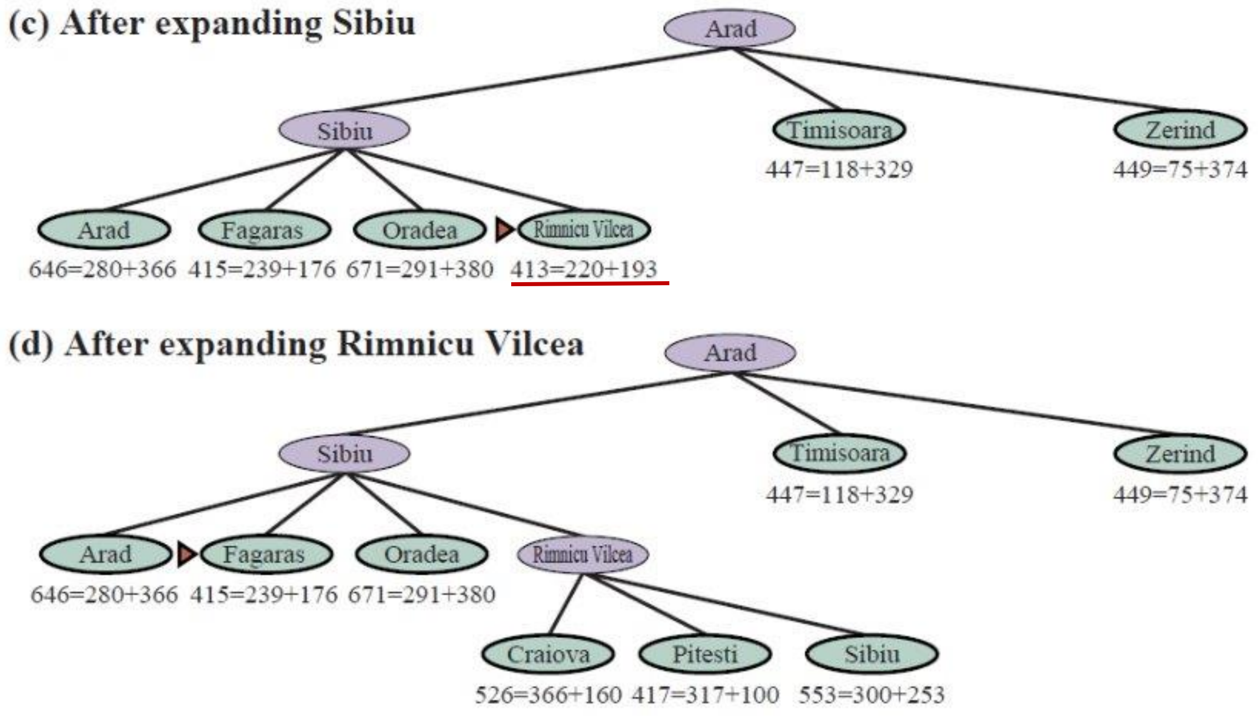
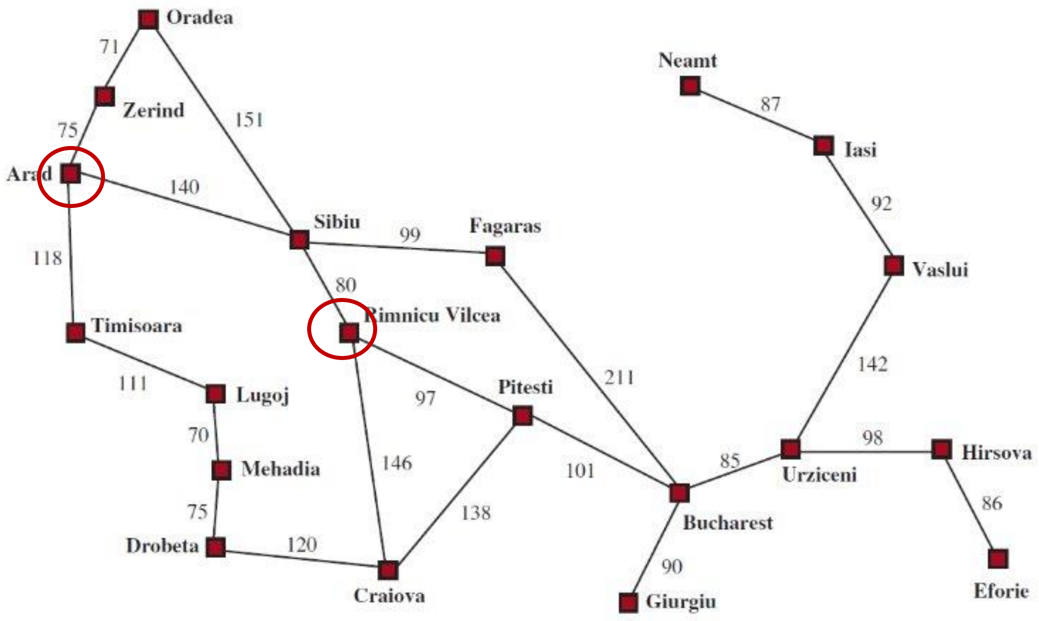
Optimal Search

A* search



(c) After expanding Sibiu

(d) After expanding Rimnicu Vilcea



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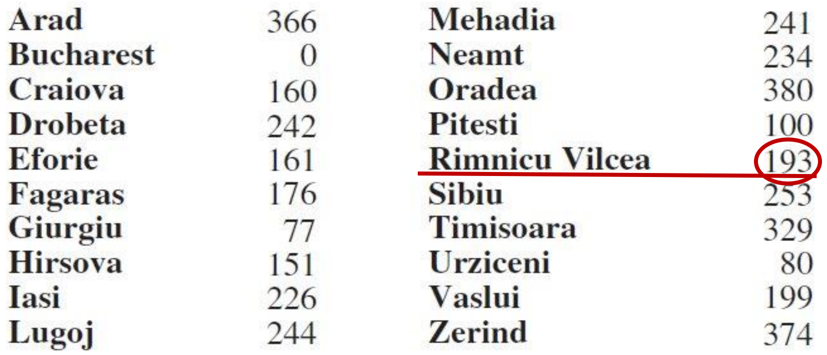

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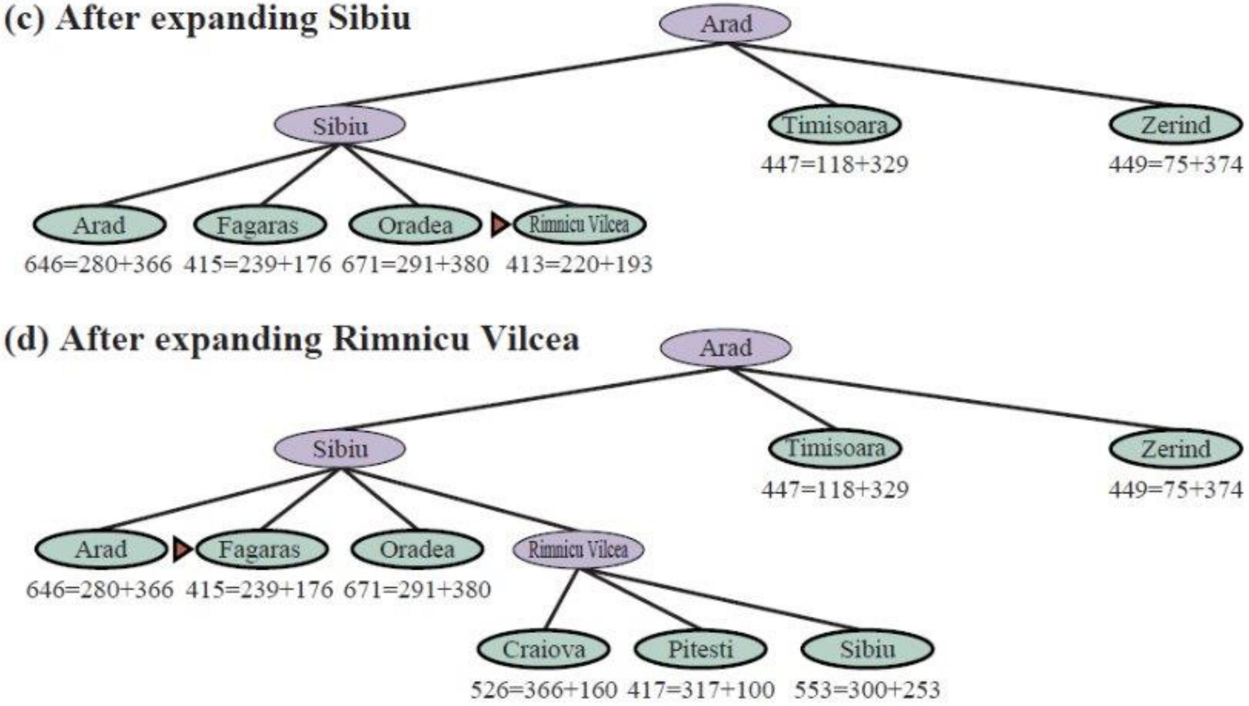
Optimal Search

A* search

- Dis. from Arad to Rimicu Vilcea: g(Rimicu) = 140 + 80 = 220
- h(Rimicu) = 193
- f(Rimicu) = 220+193 = 413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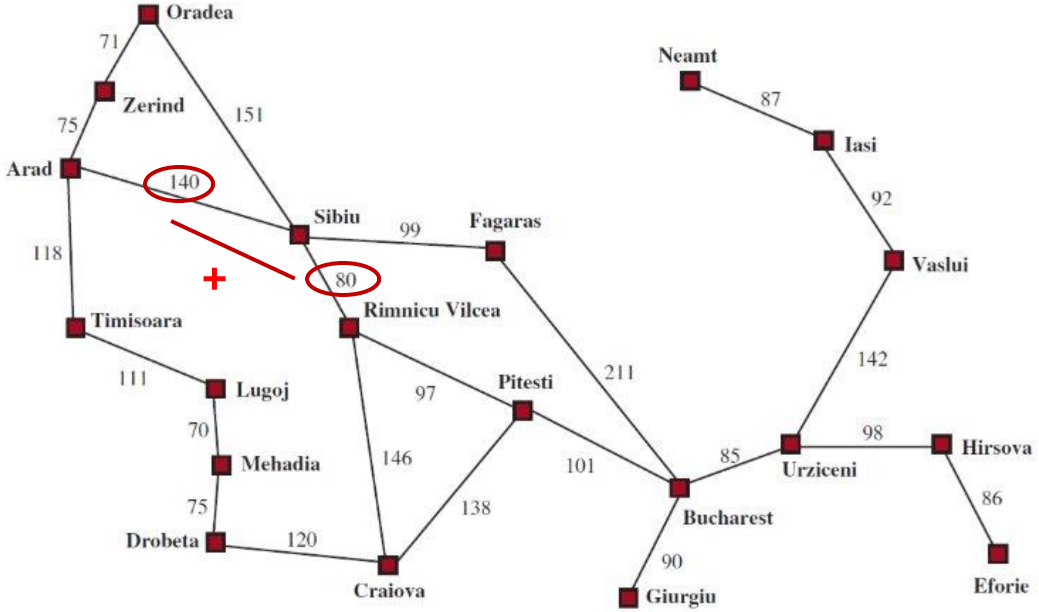

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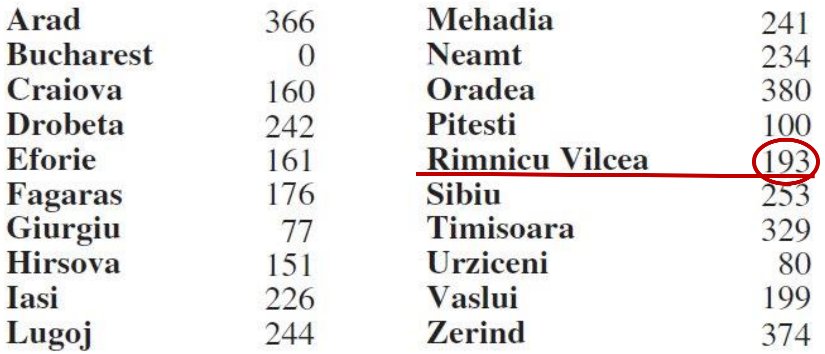

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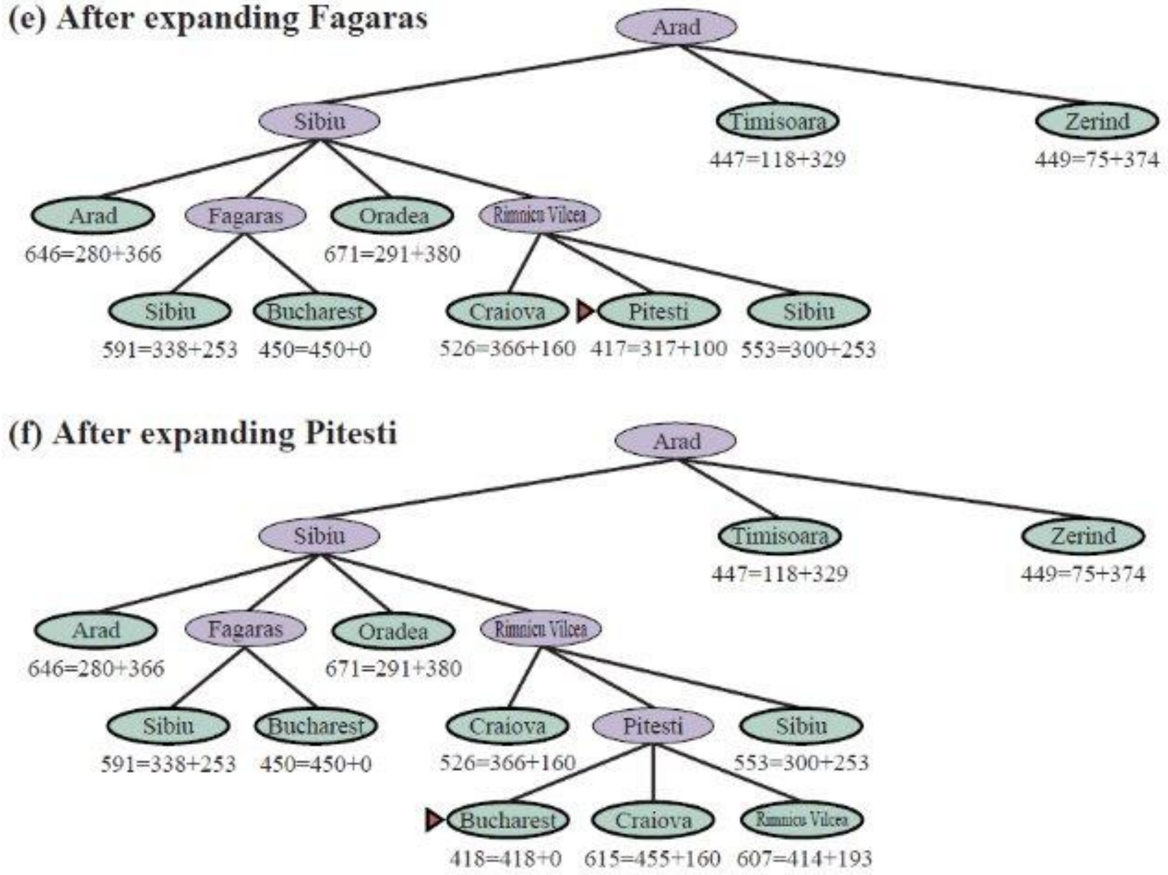
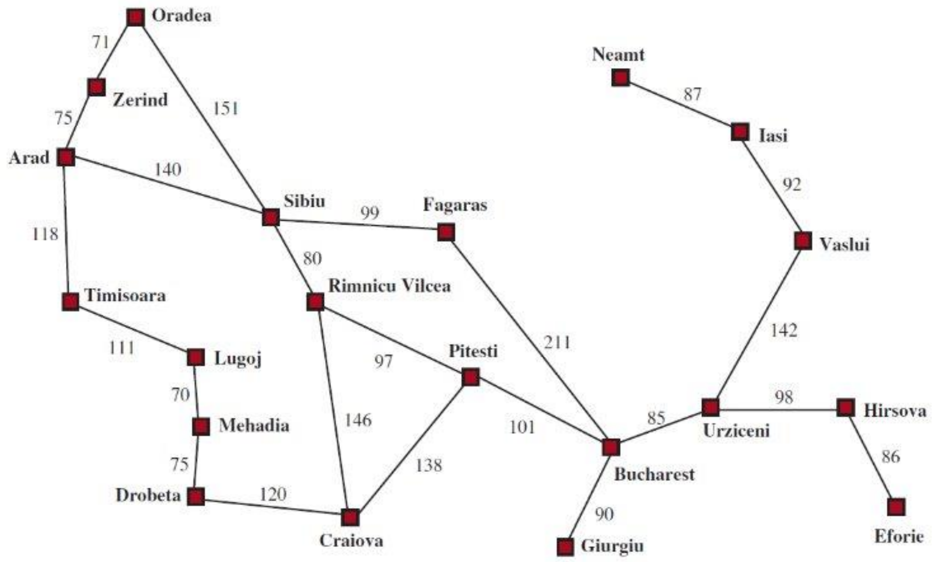
- **Alway cal. cost from initial state**

Optimal Search

A* search



Path cost for the **optimal** solution =
140+80+97+101 = 418 miles



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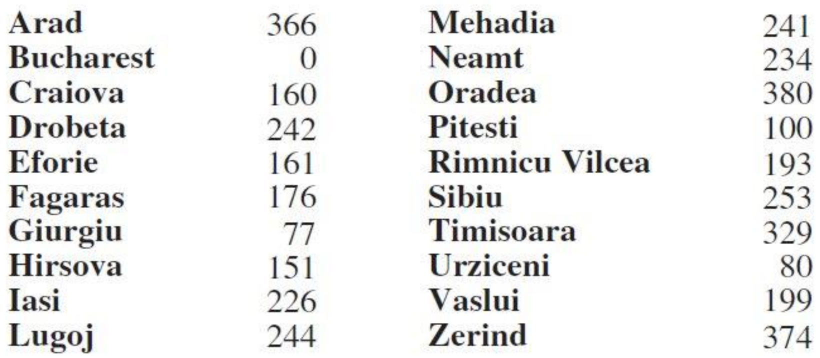

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Optimal Search

A* search for 8-puzzle

$$f(n) = g(n) + h(n)$$

0+4

2	8	3
1	6	4
7		5

2	8	3
1	6	4
7		5

Initial State

1	2	3
8		4
7	6	5

Goal State

Optimal Search

A* search for 8-puzzle

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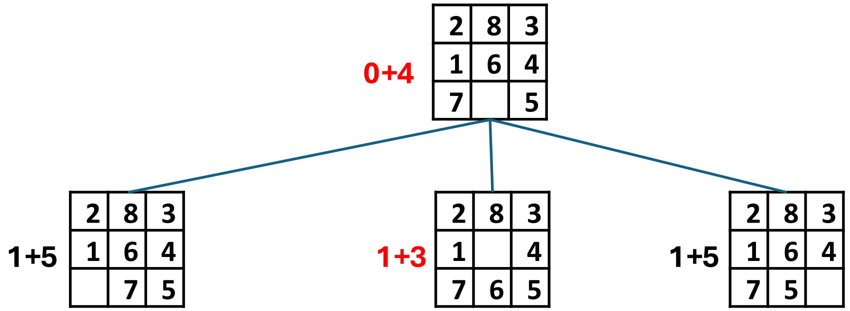

0+4

2	8	3
1	6	4
7		5

1+5

2	8	3
1	6	4
	7	5

1+3

2	8	3
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1+5

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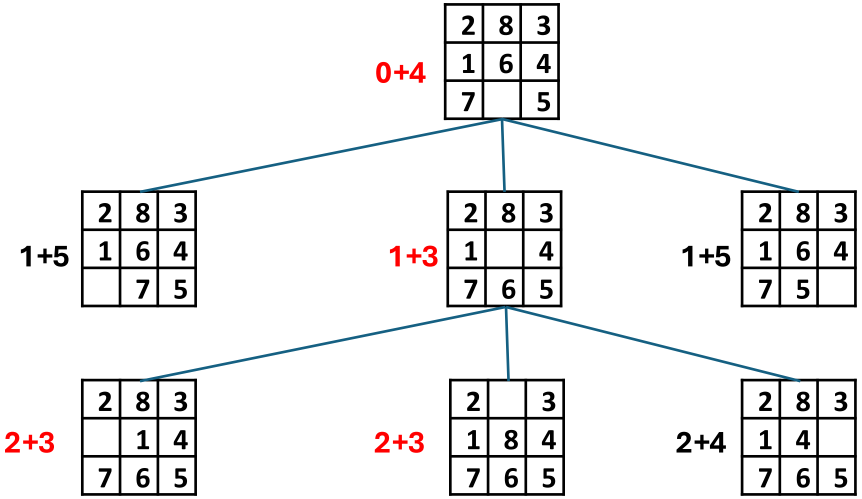

$$f(n) = g(n) + h(n)$$

Optimal Search

A* search for 8-puzzle

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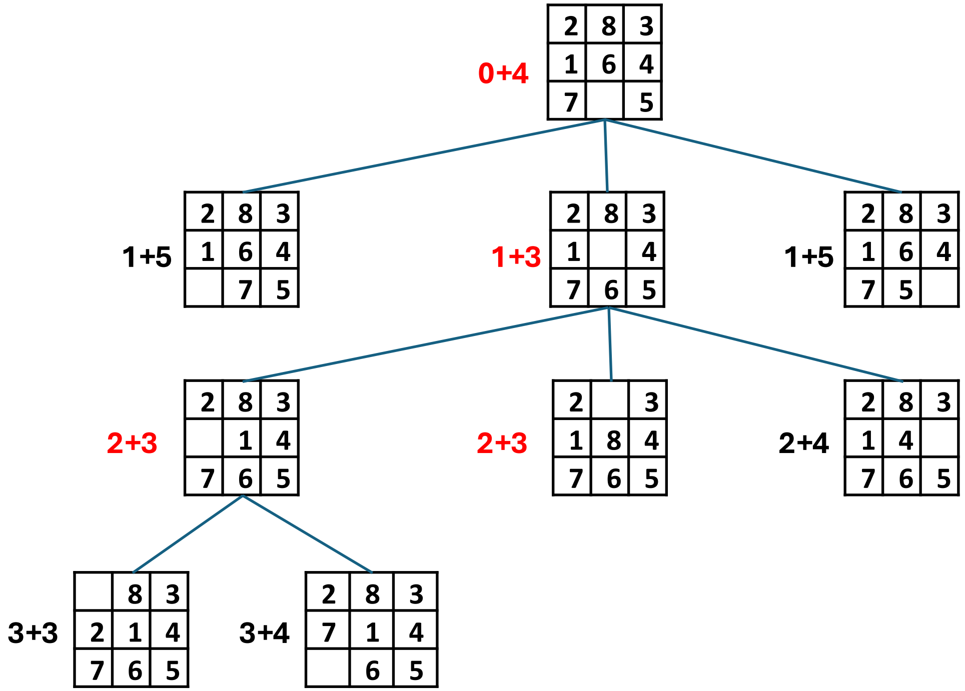

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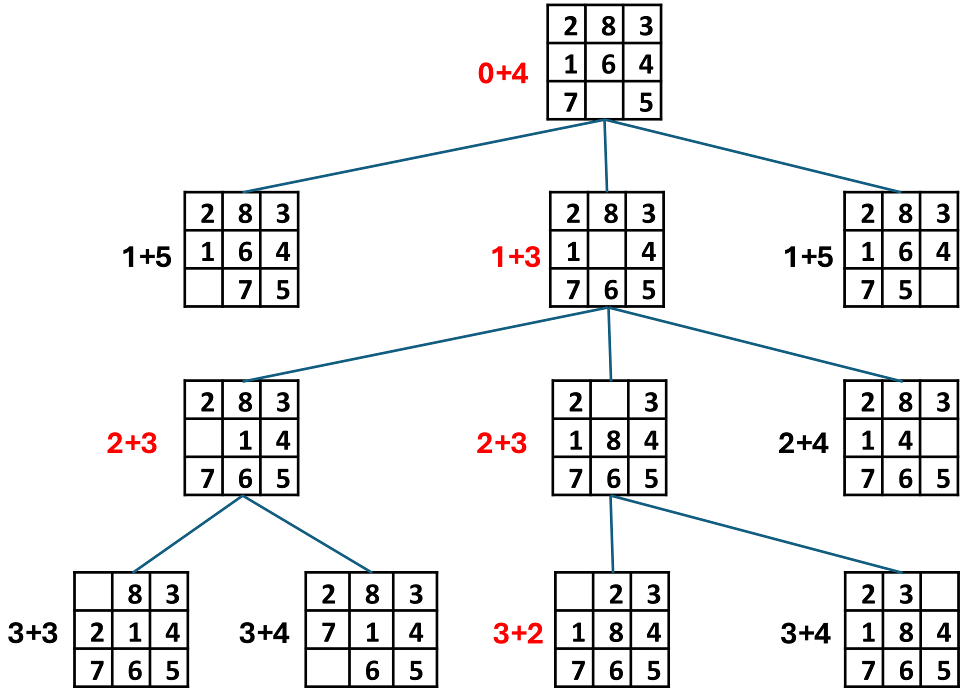

Optimal Search

A* search for 8-puzzle

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7		5

Initial State

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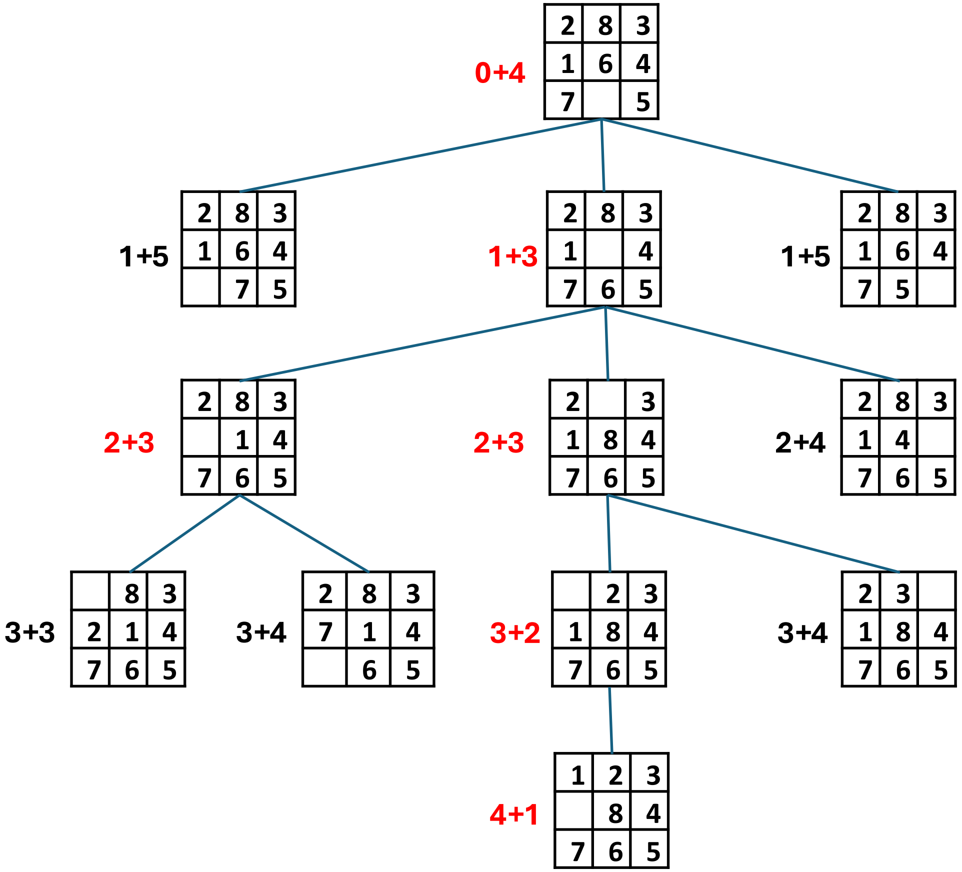
8		4
7	6	5

Goal State



0+4

2	8	3
1	6	4
7		5

1+5

2	8	3
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	7	5

1+3

2	8	3
1		4
7	6	5

1+5

2	8	3
1	6	4
7	5	

2+3

2	8	3
	1	4
7	6	5

2+3

2		3
1	8	4
7	6	5

2+4

2	8	3
1	4	
7	6	5

3+3

	8	3
2	1	4
7	6	5

3+4

2	8	3
7	1	4
	6	5

3+2

	2	3
1	8	4
7	6	5

3+4

2	3	
1	8	4
7	6	5

4+1

1	2	3
	8	4
7	6	5

Optimal Search

A* search for 8-puzzle

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Initial State

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Goal State

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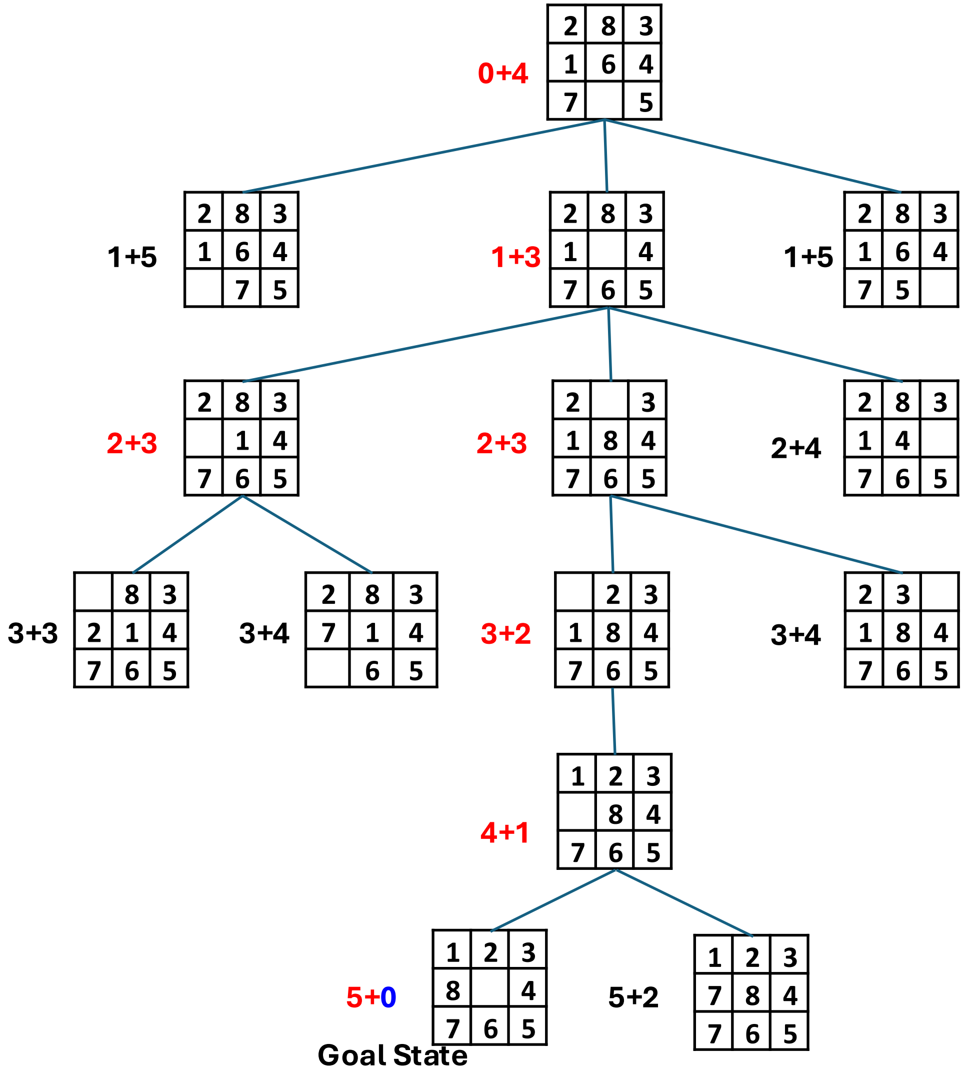
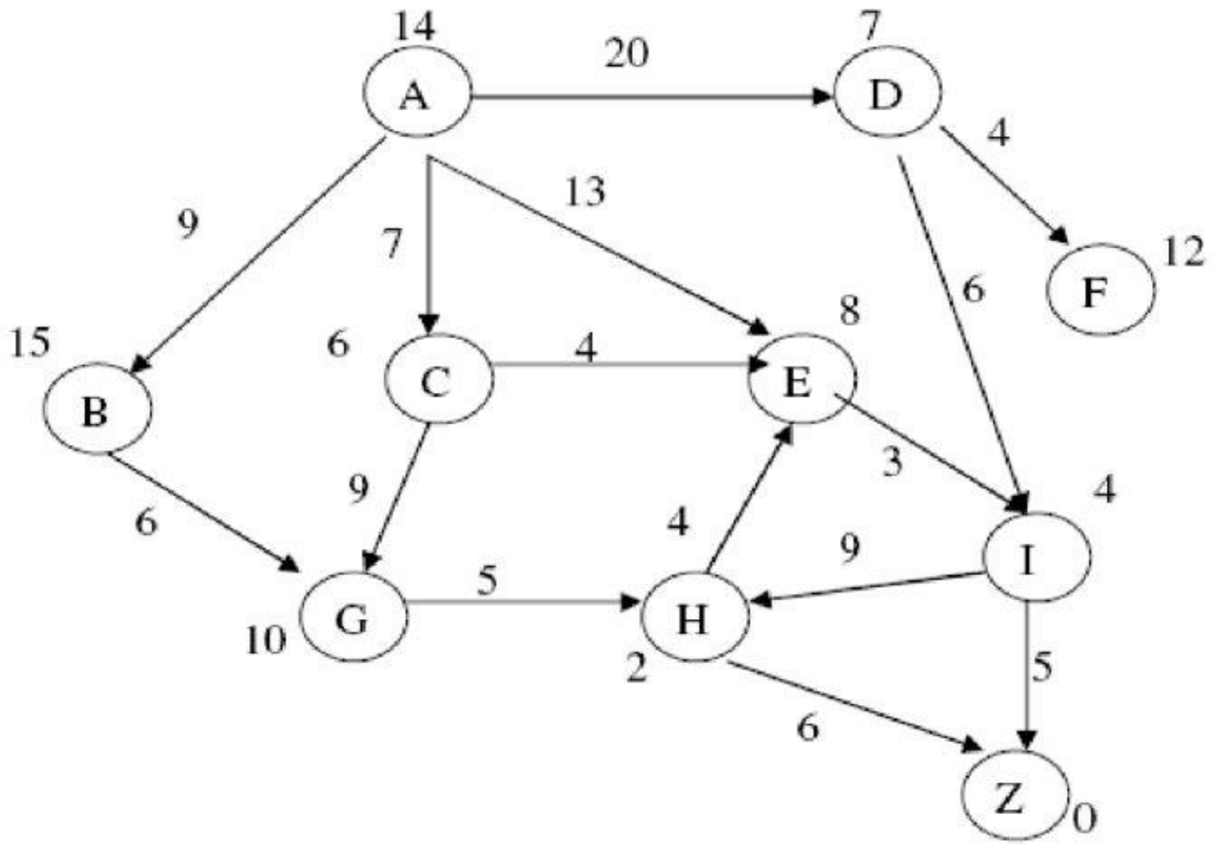


Optimal Search

A* search: Find the shortest path from A to Z using A*

- The value attached to each vertex is h(u).
- The value attached to each edge is the cost to change state k(u,v).
- Note: when g+h are the same, the **smaller g** is preferred.



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Optimal Search

Properties of Heuristic Function

- admissibility

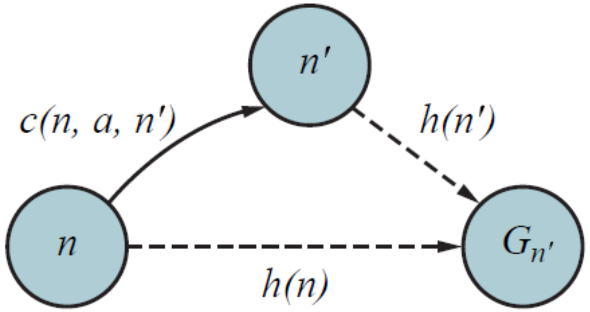
$$h(n) \leq h^{*}(n)$$

- consistency (monotonicity)

$$h(n) \leq c(n, a, n') + h(n')$$

- dominancy: h_2 **dominates** h_1

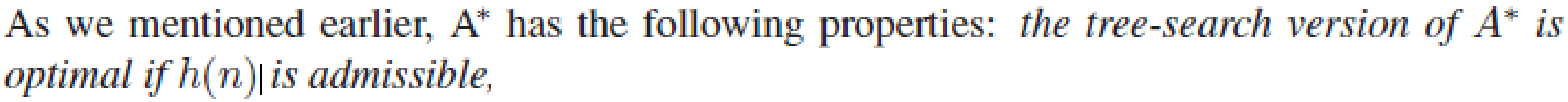
$$h_2(n) \geq h_1(n), \text{ for any node } n$$



Optimal Search

Optimality of A* algorithm

- A* is optimal if it uses an admissible(consistent) heuristic



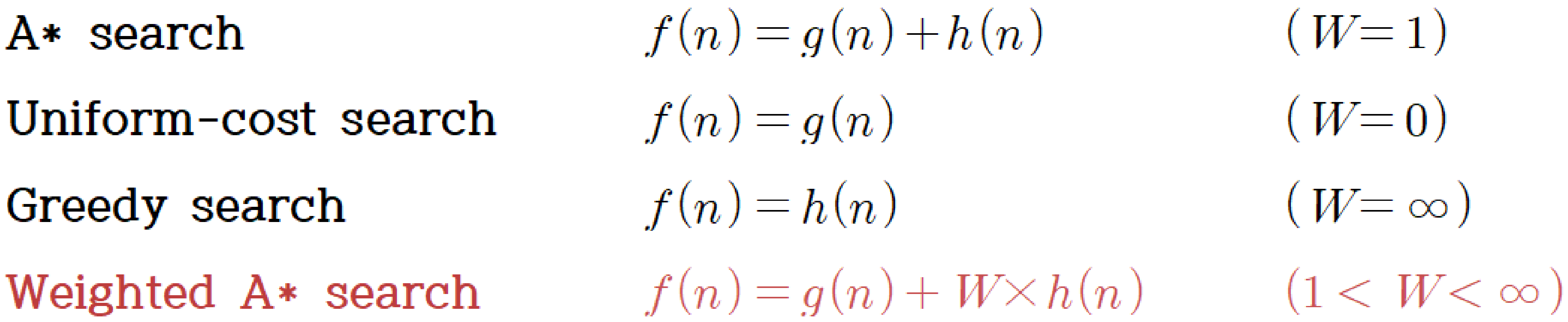
As we mentioned earlier, A* has the following properties: *the tree-search version of A* is optimal if $h(n)$| is admissible,*

Efficiency of A* algorithm

- A* with $h_2(n)$ is more efficient than A* with $h_1(n)$, if h_2 dominates h_1

Optimal Search

Weighted A* search



A* search	$f(n) = g(n) + h(n)$	$(W = 1)$
Uniform-cost search	$f(n) = g(n)$	$(W = 0)$
Greedy search	$f(n) = h(n)$	$(W = \infty)$
Weighted A* search	$f(n) = g(n) + W \times h(n)$	$(1 < W < \infty)$

- inadmissible heuristic → risk of missing optimal solution

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Weighted A* search

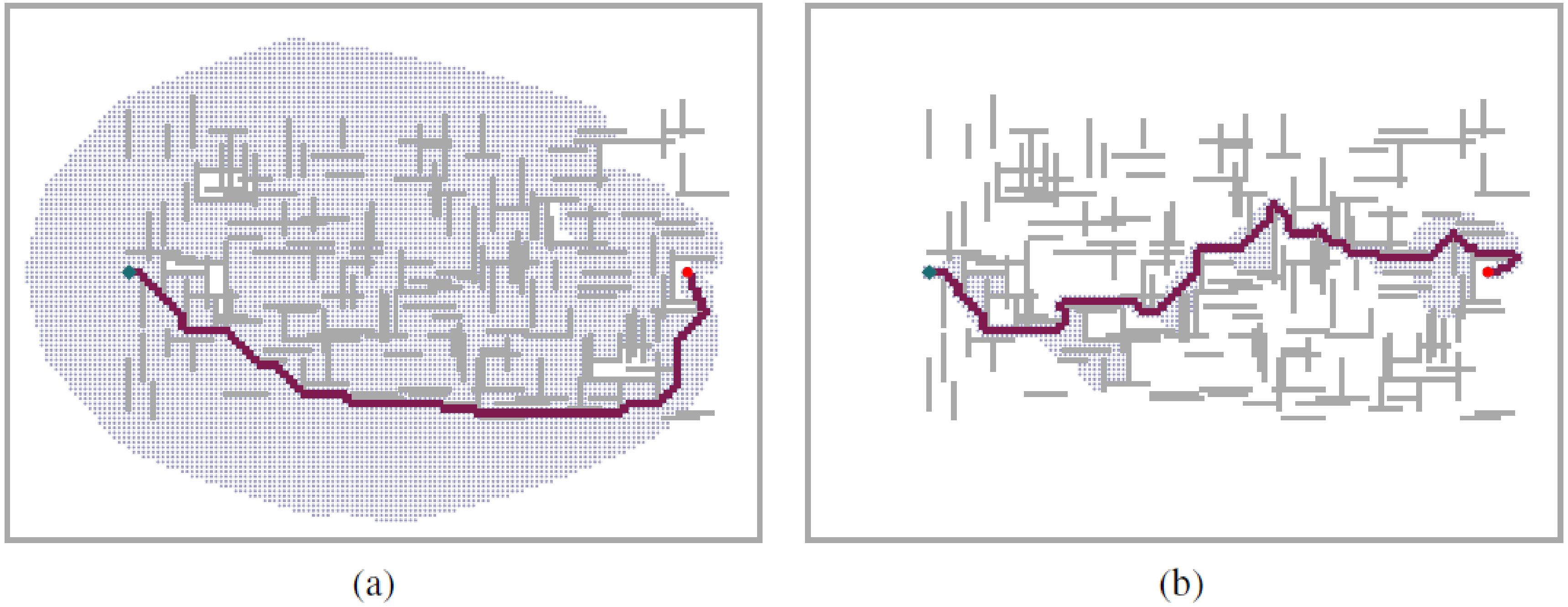

Figure 3.21 Two searches on the same grid: (a) an A* search and (b) a weighted A* search with weight $W = 2$. The gray bars are obstacles, the purple line is the path from the green start to red goal, and the small dots are states that were reached by each search. On this particular problem, weighted A* explores 7 times fewer states and finds a path that is 5% more costly.

Thank you!

You're now ready to explore the exciting world of AI!